Isomorphisms of Some Cyclic Abelian Covers of Symmetric Digraphs II

Hirobumi MIZUNO
Department of Electronics and Computer Science, Meisei University, 2-590, Nagabuti, Ome, Tokyo 198-8655, JAPAN.

and

Iwao SATO*
Oyama National College of Technology, Oyama, Tochigi 323-0806, JAPAN.

Abstract

Let $D$ be a connected symmetric digraph, $\mathbb{Z}_p$ a cyclic group of prime order $p(> 3)$ and $\Gamma$ a group of automorphisms of $D$. We enumerate the number of $\Gamma$-isomorphism classes of $g$-cyclic $\mathbb{Z}_p^3$-covers of $D$ for any nonunit $g \in \mathbb{Z}_p^3$, where $\mathbb{Z}_p^3$ is the direct sum of three $\mathbb{Z}_p$.

1. Introduction

Graphs and digraphs treated here are finite and simple. Let $D$ be a symmetric digraph and $A$ a finite group. A function $\alpha : A(D) \rightarrow A$ is called alternating if $\alpha(y, x) = \alpha(x, y)^{-1}$ for each $(x, y) \in A(D)$. For $g \in A$, a $g$-cyclic $A$-cover (or $g$-cyclic cover) $D_g(\alpha)$ of $D$ is the digraph as follows:

$$V(D_g(\alpha)) = V(D) \times A,$$

and $((u, h), (v, k)) \in A(D_g(\alpha))$ if and only if $(u, v) \in A(D)$ and $k^{-1}h\alpha(u, v) = g$.

The natural projection $\pi : D_g(\alpha) \rightarrow D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph $D'$ is called a cyclic $A$-cover of $D$ if $D'$ is a $g$-cyclic $A$-cover of $D$ for some $g \in A$. In the case that $A$ is abelian, then $D_g(\alpha)$ is simply called a cyclic abelian cover.

Let $\alpha$ and $\beta$ be two alternating functions from $A(D)$ into $A$, and let $\Gamma$ be a subgroup of the automorphism group $Aut D$ of $D$, denoted $\Gamma \leq Aut D$. Let $g, h \in A$. Then two cyclic $A$-covers $D_g(\alpha)$ and $D_h(\beta)$ are called $\Gamma$-isomorphic, denoted $D_g(\alpha) \cong_\Gamma D_h(\beta)$, if there exist an isomorphism $\Phi : D_g(\alpha) \rightarrow D_h(\beta)$ and a $\gamma \in \Gamma$ such that $\pi \Phi = \gamma \pi$, i.e., the diagram

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commutes. Let $I = \{1\}$ be the trivial subgroup of automorphisms.

Cheng and Wells [1] discussed isomorphism classes of cyclic triple covers (1-cyclic $\mathbb{Z}_3$-covers) of a complete symmetric digraph. Mizuno and Sato [16,18] enumerated the number of $I$-isomorphism classes of $g$-cyclic $\mathbb{Z}_p^n$-covers and $g$-cyclic $\mathbb{Z}_p$-covers, and $\Gamma$-isomorphism classes of $g$-cyclic $\mathbb{Z}_p$-covers of a connected symmetric digraph $D$ for any prime $p(> 2)$. Furthermore, Mizuno, Lee and Sato [15] gave a formula for the number of $I$-isomorphism classes of connected $g$-cyclic $\mathbb{Z}_p^n$-covers and connected $g$-cyclic $\mathbb{Z}_p$-covers of $D$ for any prime $p(> 2)$. Mizuno and Sato [16] gave a formula for the enumeration of $r$-isomorphism classes of $g$-cyclic $\mathbb{Z}_p \times \mathbb{Z}_p$-covers of $D$ for any prime $p(> 2)$.

Let $G$ be a graph and $A$ a finite group. Let $D(G)$ be the arc set of the symmetric digraph corresponding to $G$. Then a mapping $\alpha : D(G) \rightarrow A$ is called an ordinary voltage assignment if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The ordinary derived graph $G^\alpha$ derived from an ordinary voltage assignment $\alpha$ is defined as follows:

$$V(G^\alpha) = V(G) \times A, \text{ and } ((u, h), (v, k)) \in D(G^\alpha) \text{ if and only if } (u, v) \in D(G) \text{ and } k = h\alpha(u, v).$$

The graph $G^\alpha$ is called an $A$-covering of $G$. The $A$-covering $G^\alpha$ is an $|A|$-fold regular covering of $G$. Every regular covering of $G$ is an $A$-covering of $G$ for some group $A$ (see [3]). Furthermore the 1-cyclic $A$-cover $D_1(\alpha)$ of a symmetric digraph $D$ can be considered as the $A$-covering $D^\alpha$ of the underlying graph $\tilde{D}$ of $D$.

A general theory of graph coverings is developed in [4]. $\mathbb{Z}_2$-coverings (double coverings) of graphs were dealt in [5] and [22]. Hofmeister [6] and, independently, Kwak and Lee [12] enumerated the $I$-isomorphism classes of $n$-fold coverings of a graph, for any $n \in \mathbb{N}$. Dresbach [2] obtained a formula for the number of strong isomorphism classes of regular coverings of graphs with voltages in finite fields. The $I$-isomorphism classes of regular coverings of graphs with voltages in finite dimensional vector spaces over finite fields were enumerated by Hofmeister [7]. Hong, Kwak and Lee [9] gave the number of $I$-isomorphism classes of $\mathbb{Z}_n$-coverings, $\mathbb{Z}_p \oplus \mathbb{Z}_p$-coverings and $D_n$-coverings, $n$:odd, of graphs, respectively. Mizuno and Sato [19,20,21] presented the numbers of $\Gamma$-isomorphism classes of $\mathbb{Z}_p^n$-coverings of graphs for $n = 1, 2, 3$ and any prime $p(> 2)$.

In the case of connected coverings, Kwak and Lee [14] enumerated the $I$-isomorphism classes of connected $n$-fold coverings of a graph $G$. Furthermore, Kwak, Chun and Lee [13] gave some formulas for the number of $I$-isomorphism classes of connected $A$-coverings of $G$ when $A$ is a finite abelian group or $D_n$.

We present the number of $\Gamma$-isomorphism classes of $g$-cyclic $\mathbb{Z}_p^3$-covers of connected symmetric digraphs for any element $g \neq 0 \in \mathbb{Z}_p^3$, where $0$ is the unit of $\mathbb{Z}_p^3$ and any prime $p(> 3)$. 
2. Isomorphisms of cyclic $\mathbb{Z}_3^3$-covers

Let $D$ be a connected symmetric digraph and $A$ a finite abelian group. The group $\Gamma$ of automorphisms of $D$ acts on the set $C(D)$ of alternating functions from $A(D)$ into $A$ as follows:

$$\alpha^\gamma(x, y) = \alpha(\gamma(x), \gamma(y)) \text{ for all } (x, y) \in A(D),$$

where $\alpha \in C(D)$ and $\gamma \in \Gamma$.

Let $G$ be the underlying graph of $D$. The set of ordinary voltage assignments of $G$ with voltages in $A$ is denoted by $C^0(G; A)$. Note that $C(D) = C^1(G; A)$. Furthermore, let $C^0(G; A)$ be the set of functions from $V(G)$ into $A$. We consider $C^0(G; A)$ and $C^1(G; A)$ as additive groups. The homomorphism $\delta : C^0(G; A) \rightarrow C^1(G; A)$ is defined by $(\delta s)(x, y) = s(x) - s(y)$ for $s \in C^0(G; A)$ and $(x, y) \in A(D)$. The 1-cohomology group $H^1(G; A)$ with coefficients in $A$ is defined by $H^1(G; A) = C^1(G; A)/\text{Im } \delta$. For each $\alpha \in C^1(G; A)$, let $[\alpha]$ be the element of $H^1(G; A)$ which contains $\alpha$.

The automorphism group $\text{Aut } A$ acts on $C^0(G; A)$ and $C^1(G; A)$ as follows:

$$(\sigma s)(x) = \sigma(s(x)) \text{ for } x \in V(D),$$

$$(\sigma \alpha)(x, y) = \sigma(\alpha(x, y)) \text{ for } (x, y) \in A(D),$$

where $s \in C^0(G; A)$, $\alpha \in C^1(G; A)$ and $\sigma \in \text{Aut } A$. A finite group $B$ is said to have the isomorphism extension property (IEP), if every isomorphism between any two isomorphic subgroups $\mathcal{E}_1$ and $\mathcal{E}_2$ of $B$ can be extended to an automorphism of $B$ (see [9]). For example, the cyclic group $\mathbb{Z}_n$ for $n \in \mathbb{N}$, the dihedral group $D_n$ for odd $n \geq 3$, and the direct sum of $m$ copies of $\mathbb{Z}_p$ ($p$: prime) have the IEP.

Mizuno and Sato [18] gave a characterization for two cyclic $A$-covers of $D$ to be $\Gamma$-isomorphic.

**Theorem 1** (18, Corollary 3) Let $D$ be a connected symmetric digraph, $G$ the underlying graph of $D$, $A$ a finite abelian group with the IEP, $g \in A$, $\alpha, \beta \in C(D)$ and $\Gamma \leq \text{Aut } D$. Assume that the order of $g$ is odd. Then the following are equivalent:

1. $D_g(\alpha) \cong \Gamma D_g(\beta)$.
2. There exist $\gamma \in \Gamma$, $\sigma \in \text{Aut } A$ and $s \in C^0(G; A)$ such that

$$\beta = \sigma \alpha^\gamma + \delta s \text{ and } \sigma(g) = g.$$

Let $\text{Iso}(D, A, g, \Gamma)$ denote the number of $\Gamma$-isomorphism classes of $g$-cyclic $A$-covers of $D$. The following result holds.

**Theorem 2** (18, Theorem 3) Let $D$ be a connected symmetric digraph, $G$ the underlying graph of $D$, $A$ a finite abelian group with the IEP, $g, h \in A$ and $\Gamma \leq \text{Aut } D$. Assume that the orders of $g$ and $h$ are equal and odd, and $\rho(g) = h$ for some $\rho \in \text{Aut } A$. Then

$$\text{Iso}(D, A, g, \Gamma) = \text{Iso}(D, A, h, \Gamma).$$
Let \( p(> 3) \) be prime and \( \mathbb{Z}_p \) the cyclic group of order \( p \). Then \( \mathbb{Z}_p^3 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \) has the IEP. Since \( \mathbb{Z}_p^3 \) is the 3-dimensional vector space over \( \mathbb{Z}_p \), the general linear group \( GL_3(\mathbb{Z}_p) \) is the automorphism group of \( \mathbb{Z}_p^3 \). Furthermore, \( GL_3(\mathbb{Z}_p) \) acts transitively on \( \mathbb{Z}_p^3 \setminus \{0\} \). Set \( e = e_1 = (100)^t \in \mathbb{Z}_p^3 \).

By Theorem 2, we have \( Iso(D, A, g, \Gamma) = Iso(D, A, e, \Gamma) \) for any element \( g \in \mathbb{Z}_p^3 \setminus \{0\} \). Thus we consider the number of \( \Gamma \)-isomorphism classes of \( e \)-cyclic \( \mathbb{Z}_p^3 \)-covers of \( D \).

Let \( \Gamma \leq Aut \ D \) and \( \Pi = GL_3(\mathbb{Z}_p) \). Furthermore, set \( \Pi_e = \{ \sigma \in \Pi \mid \sigma(e) = e \} \).

An action of \( \Pi_e \times \Gamma \) on \( H^1(G; \mathbb{Z}_p^3) \) is defined as follows:

\[
(A, \gamma)[\alpha] = [A\alpha\gamma] = \{A\alpha\gamma + \delta s \mid s \in C^0(G; \mathbb{Z}_p^3)\},
\]

where \( A \in \Pi_e \), \( \gamma \in \Gamma \) and \( \alpha \in C^1(G; \mathbb{Z}_p^3) \). By Theorem 1, the number of \( \Gamma \)-isomorphism classes of \( e \)-cyclic \( \mathbb{Z}_p^3 \)-covers of \( D \) is equal to that of \( \Pi_e \times \Gamma \)-orbits on \( H^1(G; \mathbb{Z}_p^3) \).

Let \( \lambda \in \mathbb{Z}_p^* \) and \( i \) an integer. Then we introduced two types of matrices as follows:

\[
D_{n, \lambda} = \begin{bmatrix}
\lambda & 0 & 0 & \cdots & 0 & 0 \\
1 & \lambda & 0 & \cdots & 0 & 0 \\
0 & 1 & \lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & \lambda \\
0 & 0 & 0 & \cdots & 1 & \lambda
\end{bmatrix},
\]

where \( 0 \leq i < p^2 - 1, i \equiv 0 \pmod{p + 1} \).

Let \( K = GF(p^2) \) be the finite field with \( p^2 \) elements, and \( \rho \) be a generator of the cyclic multiplicative group \( K^* \). Then, identifying \( GL_2(\mathbb{Z}_p) \) with \( GL(K) \), the matrix \( B_2 \) is defined as follows:

\[
B_2(\tau) = \rho \tau, \tau \in K \ (c.f., [11]).
\]

Note that \( ord(B_2) = p^2 - 1 \), where \( ord(\lambda) \) is the order of \( \lambda \).

Let \( D \) be a connected symmetric digraph, \( G \) the underlying graph of \( D \), \( \Gamma \leq Aut \ D \), \( \gamma \in \Gamma \), \( \lambda \in \mathbb{Z}_p^* \) and \( 0 \leq i < p^2 - 1 \) such that \( i \equiv 0 \pmod{p + 1} \).

A \( \gamma \)-orbit \( \sigma \) of length \( k \) on \( E(G) \) is called diagonal if \( \sigma = \{x, \gamma^k(x)\} \) for some \( x \in V(G) \). The vertex orbit < \( \gamma \) > \( x \) and the arc orbit < \( \gamma \) > \( (x, \gamma^k(x)) \) are also called diagonal.

Let \( z = \lambda, B_2 \) and \( m = ord(z) \). Then a diagonal arc orbit of length \( 2k \) (the corresponding edge orbit of length \( k \) and the corresponding vertex orbit of length \( 2k \)) is called type-1 if \( z^k = -1 \) or \( z^k = -I \), and type-2 otherwise.

Let \( k \in \mathbb{N} \). A < \( \gamma \)-orbit \( \sigma \) on \( V(G) \), \( E(G) \) or \( D(G) \) is called \( k \)-divisible if \( |\sigma| \equiv 0 \pmod{k} \).

A < \( \gamma \)-orbit \( \sigma \) on \( V(G) \) is called edge-induced if there exists a orbit < \( \gamma \> \{x, y\} \) on \( E(G) \) with \( x, y \in \sigma \). A \( k \)-divisible < \( \gamma \>-orbit \( \sigma \) on \( V(G) \) is called strongly \( k \)-divisible if \( \sigma \) is edge-induced and satisfies the following condition:

If \( \Omega = < \gamma > (x, y) \) is any < \( \gamma \>-orbit on \( D(G) \), and \( y = \gamma^j(x), x, y \in \sigma \), then \( j \equiv 0 \pmod{k} \).
Note that, if $\sigma = \langle \gamma \rangle > x$ is strongly $m$-divisible, $|\sigma| = t$ and there exists a diagonal $\langle \gamma \rangle$-orbit $\Omega = \langle \gamma \rangle \cdot (x, \gamma^{t/2}(x))$ on $D(G)$, then $\Omega$ is type-2.

For $\gamma \in \Gamma$, let $G(\gamma)$ be a simple graph whose vertices are the $\langle \gamma \rangle$-orbits on $V(G)$, with two vertices adjacent in $G(\gamma)$ if and only if some two of their representatives are adjacent in $G$. The $k$th $p$-level of $G(\gamma)$ is the induced subgraph of $G(\gamma)$ on the vertices $\omega$ such that $\theta(\omega) = p^k$, where $\theta(i)$ is the largest power of $p$ dividing $i$. A $p$-level component of $G(\gamma)$ is a connected component of some $p$-level of $G(\gamma)$.

Let $z = A, B$, and $m = \text{ord}(z)$. Then, let $G_z(\gamma)$ be the subgraph induced by the set of $m$-divisible $\langle \gamma \rangle$-orbits on $V(G)$. Note that $G_1(\gamma) = G(\gamma)$ for $z = \lambda = 1$. The $k$th $p$-level and $p$-level components of $G_z(\gamma)$ are defined similarly to the case of $G(\gamma)$. A $p$-level component $K$ of $G_z(\gamma)$ is called minimal if each $\sigma$ of $K$ satisfies the condition $\theta(\sigma) < \theta(\omega)$ whenever $\omega \notin K$ and $\sigma \omega \in E(G(\gamma))$ (c.f., [1]).

Let $H$ be a $p$-level component of $G_z(\gamma)$ (a 0th $p$-level component of $G_\lambda(\gamma)$). Then $H$ is called $z$-favorable(0-favorable) if $H$ satisfies one of the following conditions:

1. $H$ is not minimal,
2. some $\sigma$ of $H$ is not strongly $m$-divisible, or
3. some $\sigma$ of $H$ is type-1 diagonal.

Otherwise $H$ is called $z$-defective(0-defective). If $z = \lambda$, then $z$-defective($z$-favorable) $p$-level components of $G_z(\gamma)$ are defective(favorable) $p$-level components of $G_\lambda(\gamma)$ (see [21]).

Let $\lambda \in Z^*_p$ and $m = \text{ord}(\lambda)$. For $k \geq 1$, let $H$ be a $k$th $p$-level component of $G_\lambda(\gamma)$. Then $H$ is called $\lambda$-semidefective if $H$ is not $\lambda$-favorable, and some $\sigma$ of $H$ is strongly $m$-divisible but not strongly $p$-divisible. Furthermore, $H$ is called $\lambda$-strongly defective if $H$ is minimal, and each vertex of $H$ is strongly $pm$-divisible but not type-1 diagonal.

**Theorem 3** Let $D$ be a connected symmetric digraph, $G$ its underlying graph, $p(>3)$ prime, $g \in Z_p^2 \setminus \{0\}$ and $\Gamma \leq \text{Aut } D$. Let $z = \lambda, B$, where $\lambda \in Z^*_p$, $0 \leq i < p^2 - 1, i \neq 0 \pmod{p+1}$ and $m = \text{ord}(z)$. For $\gamma \in \Gamma$ and $z$, let $e(\gamma), \kappa(\gamma, z), \mu(\gamma, z), \nu(\gamma, z)$ and $d(\gamma, z)$ be the number of $\langle \gamma \rangle$-orbits on $E(G)$, not $m$-divisible, not diagonal $\langle \gamma \rangle$-orbits on $E(G)$, type-2 diagonal $\langle \gamma \rangle$-orbits on $E(G)$, $m$-divisible $\langle \gamma \rangle$-orbits on $V(G)$, and $z$-defective $p$-level component of $G_z(\gamma)$, respectively. For $\gamma \in \Gamma$ and $\lambda \in Z^*_p$, let $v_0(\gamma, \lambda), \mu_1(\gamma, \lambda)$ and $\kappa_1(\gamma, \lambda)$ be the number of $m$-divisible, not $p$-divisible $\langle \gamma \rangle$-orbits on $V(G)$, not $m$-divisible, type-1 diagonal $\langle \gamma \rangle$-orbits on $E(G)$, and not $p$-divisible, not diagonal $\langle \gamma \rangle$-orbits on $E(G)$, respectively. Furthermore, let $c(\gamma, \lambda), d_1(\gamma, \lambda), d_2(\gamma, \lambda)$ and $d_0(\gamma, \lambda)$ be the number of $p$-level components, $\lambda$-favorable $p$-level components, $\lambda$-strongly defective $p$-level components and 0-favorable $p$-level components of $G_\lambda(\gamma)$, respectively. Then the number of $\Gamma$-isomorphism classes of $g$-cyclic $Z_p^3$-covers of $D$
is
\[ Iso(D, Z^3_p, g, \Gamma) = \frac{1}{p^3(p-1)^2(p+1)|\Gamma|} \sum_{\gamma \in \Gamma} \{p^3(\nu(\gamma) - \nu(\gamma, 1) - \kappa(\gamma, 1) - \mu(\gamma, 1) + d(\gamma, 1)) \]
\[ + (p + 1)^2(p - 1)p^{2e(\gamma) - 3\nu(\gamma, 1)} p^{2\nu(\gamma, 1) + \nu_0(\gamma, 1) - 2\mu(\gamma, 1) + \kappa_1(\gamma, 1) - 2\kappa(\gamma, 1)} 
+ \mu_1(\gamma, 1) + c(\gamma, 1) - d_1(\gamma, 1) - d_2(\gamma, 1) - d_0(\gamma, 1) + d(\gamma, 1) \]
\[ + p(p - 1)^2(p + 1)p^{e(\gamma) - 3\nu(\gamma, 1) + 2\nu_0(\gamma, 1) - \mu(\gamma, 1) + 2\kappa_1(\gamma, 1) - \kappa(\gamma, 1)} 
+ 2\mu_1(\gamma, 1) + c(\gamma, 1) - d_1(\gamma, 1) + 2d_2(\gamma, 1) - d_0(\gamma, 1) \]
\[ + p^2(p - 1) \sum_{\lambda = 2}^{p - 1} \nu(\gamma, \lambda) - \nu(\gamma, 1) - 2\kappa(\gamma, \lambda) - 2\mu(\gamma, \lambda) - 2d(\gamma, \lambda) \]
\[ + p^2(p - 1)(p + 1) \sum_{\lambda = 2}^{p - 1} p^{2e(\gamma) - 2\nu(\gamma, 1) + \nu_0(\gamma, 1) - \mu(\gamma, 1) + \kappa_1(\gamma, 1) - \kappa(\gamma, 1) + \mu_1(\gamma, 1) + c(\gamma, 1)} 
- d_1(\gamma, 1) - d_2(\gamma, 1) - d_0(\gamma, 1) + d(\gamma, 1) - \nu(\gamma, 1) - \mu(\gamma, 1) + d(\gamma, 1) \]
\[ + p^2 \sum_{\lambda = 2}^{p - 1} p^{3e(\gamma) - 2\nu(\gamma, \lambda) - \nu(\gamma, 1) - 2\kappa(\gamma, \lambda) - 2\mu(\gamma, \lambda) - 2d(\gamma, \lambda)} + d(\gamma, \lambda) \]
\[ + p^2(p - 1)(p + 1) \sum_{\lambda = 2}^{p - 1} p^{2e(\gamma) - 2\nu(\gamma, \lambda) + \nu_0(\gamma, \lambda) - \mu(\gamma, \lambda) + \kappa_0(\gamma, \lambda) - \kappa(\gamma, \lambda) + \mu_1(\gamma, \lambda) + c(\gamma, \lambda)} 
- d_1(\gamma, 1) + d_2(\gamma, 1) - d_0(\gamma, 1) - \nu(\gamma, 1) - \mu(\gamma, 1) + \kappa_1(\gamma, 1) + d(\gamma, 1) \]
\[ + p^3(p + 1) \sum_{1 < \lambda < \tau < p - 1} p^{3e(\gamma) - \nu(\gamma, \lambda) - \nu(\gamma, \tau) - \nu(\gamma, 1) - \kappa(\gamma, \lambda) - \kappa(\gamma, \tau) - \kappa(\gamma, 1) - \mu(\gamma, \lambda)} 
- \mu(\gamma, \tau) + d(\gamma, \lambda) + d(\gamma, \tau) + d(\gamma, 1) \]
\[ + p^3(p - 1) \sum_{0 < i < p^2 - 1, i \not\equiv 0 \pmod{p+1}} p^{3e(\gamma) - 2\nu(\gamma, B_i) - 2\kappa(\gamma, B_i) - 2\mu(\gamma, B_i)} 
+ 2d(\gamma, B_i) - \nu(\gamma, 1) - \kappa(\gamma, 1) - \mu(\gamma, 1) + d(\gamma, 1) \}, \]
where we select only one of \(i\) and \(i'\) such that \(ip \equiv i'(\mod{p^2 - 1})\) in the last summation.

**Proof.** Let \(\Pi = GL_3(Z_p)\). By the preceding remark and Burnside's Lemma, the number of \(\Gamma\)-isomorphism classes of \(\mathbf{e}\)-cyclic \(Z^3_p\)-covers of \(D\) is
\[
\frac{1}{|\Pi_e|} \cdot |\Gamma| \sum_{(A, \gamma) \in \Pi_e \times \Gamma} |H^1(G; Z^3_p)^{(A, \gamma)}|,
\]
where \(U^{(A, \gamma)}\) is the set consisting of the elements of \(U\) fixed by \((A, \gamma)\).

Now, we have
\[
\Pi_e = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & B \end{bmatrix} \mid a, b = 0, 1, \ldots, p - 1; B \in GL_2(Z_p) \right\},
\]
Now, there exist $p^2 + p - 1$ conjugacy classes of $\Pi_e$. By Theorem 4, the representatives of these conjugacy classes are given as follows:

$$A_1 = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} tD_{2,1} \\ 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 \\ D_{2,1} \end{bmatrix}, A_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$A_{5,\lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}, (2 \leq \lambda \leq p-1), A_{6,\lambda} = \begin{bmatrix} tD_{2,1} \\ \lambda \end{bmatrix}, (2 \leq \lambda \leq p-1),$$

$$A_{7,\lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, (2 \leq \lambda \leq p-1), A_{8,\lambda} = \begin{bmatrix} 1 \\ D_{2,\lambda} \end{bmatrix}, (2 \leq \lambda \leq p-1),$$

$$A_{9,\lambda,\tau} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \tau \end{bmatrix}, (2 \leq \lambda \neq \tau \leq p-1), A_{10,\iota} = \begin{bmatrix} 1 \\ B_2 \end{bmatrix}, (0 < \iota < p^2-1, \iota \neq 0 (mod p+1)).$$

where we select only one of $\iota$ and $\iota'$ such that $\iota p \equiv \iota'(mod p^2-1)$ for $B_2$ (see [11]). The cardinalities of the first, second, · · · , tenth type of conjugacy classes are as follows:

$$1, p^2 - 1, p(p - 1)(p + 1), p(p - 1)^2(p + 1), p^2(p + 1), p^2(p - 1)(p + 1),$$

$$p^2, p^2(p - 1)(p + 1), p^3(p + 1), p^3(p - 1).$$

Furthermore, the number of the first, second, · · · , tenth type of conjugacy classes are as follows:

$$1, 1, 1, 1, p - 2, p - 2, p - 2, p - 2, \frac{1}{2}(p - 2)(p - 3), \frac{1}{2}p(p - 1).$$

The detail is developed in Section 3.

Let $A, F \in \Pi_e$ be conjugate. Then there exists an element $C \in \Pi_e$ such that $CAC^{-1} = F$. Thus $[\alpha] \in H^1(G; \mathbb{Z}_p^3)(A, \gamma)$ if and only if $A\alpha^\gamma = \alpha + \delta s$ for some $s \in C^0(G; \mathbb{Z}_p^3)$. But $A\alpha^\gamma = \alpha + \delta s$ if and only if $F(C\alpha)^\gamma = C\alpha + \delta(Cs)$, i.e., $[C\alpha] \in H^1(G; \mathbb{Z}_p^3)(F, \gamma)$. By the fact that a mapping $[\alpha] \mapsto [C\alpha]$ is bijective, we have

$$\left| H^1(G; \mathbb{Z}_p^3)(A, \gamma) \right| = \left| H^1(G; \mathbb{Z}_p^3)(F, \gamma) \right|. $$

Therefore the number of $\Gamma$-isomorphism classes of e-cyclic $\mathbb{Z}_p^3$-covers of $D$ is

$$\text{Iso}(D, \mathbb{Z}_p^3, e, \Gamma) = \frac{1}{p^2(p - 1)^2(p + 1)} \sum_{[\gamma] \in \Gamma} \left| H^1(G; \mathbb{Z}_p^3)(A_1, \gamma) \right| + (p - 1)(p + 1) \left| H^1(G; \mathbb{Z}_p^3)(A_2, \gamma) \right| + p(p - 1)(p + 1) \left| H^1(G; Z_p^3)(A_3, \gamma) \right| + p(p - 1)^2(p + 1) \times$$

$$\left| H^1(G; \mathbb{Z}_p^3)(A_4, \gamma) \right| + p^2(p + 1) \sum_{\lambda=2}^{p-1} \left| H^1(G; \mathbb{Z}_p^3)(A_5, \lambda, \gamma) \right| + p^2(p - 1)(p + 1) \sum_{\lambda=2}^{p-1} \left| H^1(G; \mathbb{Z}_p^3)(A_6, \lambda, \gamma) \right|$$

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\[ +p^2 \sum_{\lambda=2}^{p-1} |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_\lambda, \gamma)| + p^2(p-1)(p+1) \sum_{\lambda=2}^{p-1} |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_\lambda, \gamma)| \]

\[ +p^3(p+1) \sum_{2 \leq \lambda < \tau \leq p-1} |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_\lambda, \gamma)| + p^3(p-1) \sum_{\tau} |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_{10}, \gamma)|. \]

Let \((A, \gamma) \in \Pi_e \times \Gamma\).

Case 1: \(A = A_1 = 1_3\).

Then \([a] \in H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_1, \gamma)\) if and only if \(A_1 \alpha \gamma = \alpha + \delta s\) for some \(s \in C^0(G; \mathbb{Z}_p^3)\).

Now, let \(\alpha = ae_1 + be_2 + ce_3, a, b, c \in C^1(G; \mathbb{Z}_p^3),\) where \(e_1 = t(100), e_2 = t(010)\) and \(e_3 = t(001)\). Furthermore, let \(s = ve_1 + we_2 + ze_3, v, w, z \in C^0(G; \mathbb{Z}_p)\). Then \(\alpha \gamma = \alpha + \delta s\) if and only if \(a \gamma = a + \delta v, b \gamma = b + \delta w,\) and \(c \gamma = c + \delta z,\) i.e., \((1, \gamma)[a] = [a], (1, \gamma)[b] = [b]\) and \((1, \gamma)[c] = [c]\). Note that \([a], [b], [c] \in H^1(G; \mathbb{Z}_p)\). Since \([ae_1 + be_2 + ce_3] = [a]e_1 + [b]e_2 + [c]e_3,\) we have

\[ |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_1, \gamma)| = |H^1(G; \mathbb{Z}_p)^{(1, \gamma)}|^3. \]

By Theorem 3.3 of [21], it follows that

\[ |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_1, \gamma)| = p^3(e(\gamma) - r(\gamma, 1) - \kappa(\gamma, 1) - \mu(\gamma, 1)). \]

Case 2: \(A = A_2\).

Similarly to case 1, we have

\[ |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_2, \gamma)| = |H^1(G; \mathbb{Z}_p^2)^{(D_2, 1, \gamma)}| \cdot |H^1(G; \mathbb{Z}_p)^{(1, \gamma)}| . \]

But, \(D_{2, 1}\) and \(D_{2, 1}\) are conjugate in \(GL_2(\mathbb{Z}_p)\), and so

\[ |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_2, \gamma)| = |H^1(G; \mathbb{Z}_p)^{(D_2, 1, \gamma)}| . \]

By Theorem 3 of [20] and Theorem 3.3 of [21], it follows that

\[ |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_2, \gamma)| = p^{2e(\gamma) - 3v(\gamma, 1) + \nu_0(\gamma, 1) - 2\mu(\gamma, 1) + \kappa_1(\gamma, 1) - 2\kappa(\gamma, 1) + \mu_1(\gamma, 1) + c(\gamma, 1) - d_1(\gamma, 1) + d_2(\gamma, 1) - d_3(\gamma, 1) + d(\gamma, 1). \]

Case 3: \(A = A_3\).

Then we have

\[ |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_3, \gamma)| = |H^1(G; \mathbb{Z}_p)^{(1, \gamma)}| \cdot |H^1(G; \mathbb{Z}_p)^{(D_2, 1, \gamma)}| . \]

By case 2, it follows that

\[ |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_3, \gamma)| = |H^1(G; \mathbb{Z}_p)^{(A_2, \gamma)}| . \]

Case 4: \(A = A_4\).

Then \(B_3,1\) and \(A_4\) are conjugate in \(GL_3(\mathbb{Z}_p)\), and so

\[ |H^1(G; \mathbb{Z}_p\mathbb{Z}^3(\mathbf{A}_4, \gamma)| = |H^1(G; \mathbb{Z}_p)^{(D_3, 1, \gamma)}| . \]
By Theorem 3 of [20], it follows that
\[ |H^1(G; \mathbb{Z}_p^3(A_4, \gamma))| = p^{e(\gamma)} - 3\nu(\gamma, 1) + 2\nu_0(\gamma, 1) - \mu(\gamma, 1) + 2\kappa_1(\gamma, 1) - \kappa(\gamma, 1) + 2\mu_1(\gamma, 1) + c(\gamma, 1) - d_1(\gamma, 1) + 2d_0(\gamma, 1) - d_0(\gamma, 1). \]

Case 5: \( A = A_5, \gamma \).
Then we have
\[ |H^1(G; \mathbb{Z}_p^3(A_5, \lambda, \gamma))| = |H^1(G; \mathbb{Z}_p^3(A, \gamma))|^2. \]

By Theorem 3.3 of [21], it follows that
\[ |H^1(G; \mathbb{Z}_p^3(A_5, \lambda, \gamma))| = p^{3e(\gamma) - 2\nu(\gamma, 1) - \nu(\gamma, \lambda) - 2\kappa(\gamma, \lambda) - \kappa(\gamma, 1) - \mu(\gamma, 1) + 2d(\gamma, 1) + d(\gamma, \lambda)}. \]

Case 6: \( A = A_6, \lambda \).
Then we have
\[ |H^1(G; \mathbb{Z}_p^3(A_6, \lambda, \gamma))| = |H^1(G; \mathbb{Z}_p^3(D_2, 1, \gamma))| \cdot |H^1(G; \mathbb{Z}_p^3(A, \lambda, \gamma))|^2. \]

By Theorem 3 of [20] and Theorem 3.3 of [21], it follows that
\[ |H^1(G; \mathbb{Z}_p^3(A_6, \lambda, \gamma))| = p^{2e(\gamma) - 2\nu(\gamma, 1) + \nu_0(\gamma, 1) - \mu(\gamma, 1) + \kappa_1(\gamma, 1) - \kappa(\gamma, 1) + \mu_1(\gamma, 1) + c(\gamma, 1) - d_1(\gamma, 1) + d_2(\gamma, 1) - d_0(\gamma, 1) - \nu(\gamma, \lambda) - \mu(\gamma, \lambda) - \kappa(\gamma, \lambda) + d(\gamma, \lambda)}. \]

Case 7: \( A = A_7, \lambda \).
Then we have
\[ |H^1(G; \mathbb{Z}_p^3(A_7, \lambda, \gamma))| = |H^1(G; \mathbb{Z}_p^3(A, \lambda, \gamma))|^2. \]

By Theorem 3.3 of [21], it follows that
\[ |H^1(G; \mathbb{Z}_p^3(A_7, \lambda, \gamma))| = p^{3e(\gamma) - 2\nu(\gamma, 1) - \nu(\gamma, \lambda) - 2\kappa(\gamma, \lambda) - \kappa(\gamma, 1) - 2\mu(\gamma, \lambda) - \mu(\gamma, 1) + 2d(\gamma, \lambda) + d(\gamma, 1)}. \]

Case 8: \( A = A_8, \lambda \).
Then we have
\[ |H^1(G; \mathbb{Z}_p^3(A_8, \lambda, \gamma))| = |H^1(G; \mathbb{Z}_p^3(D_2, \lambda, \gamma))| \cdot |H^1(G; \mathbb{Z}_p^3(A, \lambda, \gamma))|^2. \]

By Theorem 3 of [20] and Theorem 3.3 of [21], it follows that
\[ |H^1(G; \mathbb{Z}_p^3(A_8, \lambda, \gamma))| = p^{2e(\gamma) - 2\nu(\gamma, 1) + \nu_0(\gamma, \lambda) - \mu(\gamma, \lambda) + \kappa_1(\gamma, \lambda) - \kappa(\gamma, \lambda) + \mu_1(\gamma, \lambda) + c(\gamma, \lambda) - d_1(\gamma, \lambda) + d_2(\gamma, \lambda) - d_0(\gamma, \lambda) - \nu(\gamma, 1) - \mu(\gamma, 1) - \kappa(\gamma, 1) + d(\gamma, 1)}. \]

Case 9: \( A = A_9, \lambda, \tau \).
Then we have
\[ |H^1(G; \mathbb{Z}_p^3(A_9, \lambda, \tau))| = |H^1(G; \mathbb{Z}_p^3(A, \lambda, \tau))| \cdot |H^1(G; \mathbb{Z}_p^3(A, \lambda, \gamma))| \cdot |H^1(G; \mathbb{Z}_p^3(A, \lambda, \gamma))|. \]
By Theorem 3.3 of [21], it follows that

\[ |H^1(G; \mathbb{Z}_p^3)_{(A_2, \lambda, \tau, \gamma)}| = p^{3\kappa(\gamma)} \nu(\gamma, 1) - \nu(\gamma, \lambda) - \nu(\gamma, \tau) - \kappa(\gamma, 1) - \kappa(\gamma, \lambda) - \kappa(\gamma, \tau) \]

\[-\mu(\gamma, 1) - \mu(\gamma, \lambda) - \mu(\gamma, \tau) + d(\gamma, 1) + d(\gamma, \lambda) + d(\gamma, \tau).\]

Case 10: \( A = A_{10, i}. \)

Then we have

\[ |H^1(G; \mathbb{Z}_p^3)_{(A_{10, i}, \gamma)}| = |H^1(G; \mathbb{Z}_p^3)_{(1, \gamma)}| \cdot |H^1(G; \mathbb{Z}_p^3)_{(B_2^i, \gamma)}|.\]

By Theorem 4 of [20] and Theorem 3.3 of [21], it follows that

\[ |H^1(G; \mathbb{Z}_p^3)_{(A_{10, i}, \gamma)}| = p^{3\kappa(\gamma)} - 2\nu(\gamma, B_2^i) - 2\kappa(\gamma, B_2^i) - 2\mu(\gamma, B_2^i) + 2d(\gamma, B_2^i) \]

\[-\nu(\gamma, 1) - \mu(\gamma, 1) - \kappa(\gamma, 1) + d(\gamma, 1).\]

By cases 1, 2, \ldots, 9 and 10, the result follows. Q.E.D.

**Corollary 1** Let \( D \) be a connected symmetric digraph, \( G \) its underlying graph, \( p(> 3) \) prime and \( g \in \mathbb{Z}_p^3 \). Then the number of \( I \)-isomorphism classes of \( g \)-cyclic \( \mathbb{Z}_p^3 \)-covers of \( D \) is

\[ \text{Iso}(D, \mathbb{Z}_p^3, g, I) = \frac{1}{p^3(p-1)^2(p+1)} \left\{ p^{3B} + (p+1)(p^3 - p^2 - 1)p^{2B} \right. \]

\[ + p(p^5 - p^4 - 2p^3 + p^2 + p + 1)p^B \}, \]

where \( B = B(G) = |E(G)| - |V(G)| + 1 \) is the Betti-number of \( G \).

**Proof.** Since \( I = \{1\} \), we have \( \epsilon(1) = |E(G)| \), \( \mu(1, z) = \mu_1(1, \lambda) = \kappa_1(1, \lambda) = d_1(1, \lambda) = d_0(1, \lambda) = d_2(1, \lambda) = 0 \), where \( z = \lambda, B_2^i \). Moreover, we have

\[ \nu(1, z) = \begin{cases} |V(G)| & \text{if } z = \lambda = 1, \\ 0 & \text{otherwise,} \end{cases} \]

\[ \kappa(1, z) = \begin{cases} 0 & \text{if } z = \lambda = 1, \\ |E(G)| & \text{otherwise,} \end{cases} \]

\[ \nu_0(1, \lambda) = \begin{cases} |V(G)| & \text{if } \lambda = 1, \\ 0 & \text{otherwise} \end{cases} \] and \( c(1, \lambda) = \begin{cases} 1 & \text{if } \lambda = 1, \\ 0 & \text{otherwise.} \end{cases} \]

Furthermore, we have

\[ d(1, z) = \begin{cases} 1 & \text{if } z = \lambda = 1, \\ 0 & \text{otherwise.} \end{cases} \]

Q.E.D.

This is the formula for \( r = 3 \) in [16, Theorem 4.4].
3. The conjugacy classes of \( GL_3(\mathbb{Z}_p) \)

Let \( p \) be an odd prime number. Then we consider all conjugacy classes of \( GL_3(\mathbb{Z}_p) \), where \( e = (100)^t \).

**Theorem 4** Let \( p \) be an odd prime. Then the representatives of the conjugacy classes of \( GL_3(\mathbb{Z}_p) \) are given as follows:

\[
\begin{align*}
\mathbf{A}_1 &= \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 \\ \ast & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \\
\mathbf{A}_{5,\lambda} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} (2 \leq \lambda \leq p - 1), \quad \mathbf{A}_{6,\lambda} = \begin{bmatrix} 1 & 0 \\ \ast & 1 \\ 0 & \lambda \end{bmatrix} (2 \leq \lambda \leq p - 1), \\
\mathbf{A}_{7,\lambda} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} (2 \leq \lambda \leq p - 1), \quad \mathbf{A}_{8,\lambda} = \begin{bmatrix} 1 \\ 0 \\ \lambda \end{bmatrix} (2 \leq \lambda \leq p - 1), \\
\mathbf{A}_{9,\lambda,\tau} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \tau \end{bmatrix} (2 \leq \lambda \neq \tau \leq p - 1), \quad \mathbf{A}_{10,i} = \begin{bmatrix} 1 \\ 0 \\ \mathbf{B}_2 \end{bmatrix} (0 < i < p^2 - 1, i \neq 0 (\text{mod } p + 1)).
\end{align*}
\]

**Proof.** Let \( \Pi = GL_3(\mathbb{Z}_p) \). Then we have

\[
\Pi_\alpha = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & B \end{bmatrix} \mid a, b = 0, 1, \ldots, p - 1; B \in GL_2(\mathbb{Z}_p) \right\}.
\]

Now, let

\[
\begin{align*}
\mathbf{A} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & B \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 & c & d \\ 0 & 1 & D \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} k & l \\ m & n \end{bmatrix},
\end{align*}
\]

where \( c, d = 0, 1, \ldots, p - 1; B, D \in GL_2(\mathbb{Z}_p) \). Then we have

\[
\mathbf{F}^{-1} \mathbf{A} \mathbf{F} = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & D^{-1}BD \end{bmatrix},
\]

where

\[
\alpha = c + 1 / |D| \{ -(cn - dm)(sk + tm) + (cl - dk)(uk + vm) \},
\]

\[
\beta = d + 1 / |D| \{ -(cn - dm)(sl + tn) + (cl - dk)(ul + vn) \}.
\]

Next, we consider the condition for \( B \) and \( D \) to satisfy the equation \( D^{-1}BD = B \). Suppose that \( BD = DB \). Then we have

\[
\begin{align*}
\mathbf{B} \mathbf{D} &= \begin{bmatrix} sk + tm & sl + tn \\ uk + vm & ul + vn \end{bmatrix}, \quad \mathbf{D} \mathbf{B} &= \begin{bmatrix} sk + ul & tk + vl \\ sm + un & tm + vn \end{bmatrix}.
\end{align*}
\]
If $BD = DB$, then

\[
\begin{align*}
    sk + tm &= sk + ul \\
    sl + tn &= tk + vl \\
    uk + vn &= sm + un \\
    ul + vn &= tm + vn
\end{align*}
\]

i.e.,

\[
\begin{align*}
    ul &= tm \\
    t(n-k) &= l(v-s) \\
    u(n-k) &= m(v-s)
\end{align*}
\]

Thus, the $(1, 2)$-array of $F^{-1}AF$ is

\[
c + 1/ \mid \begin{array}{c}
    c \\
    d
\end{array} \mid (lm - nk)(cs + du) = (1 - s)c - ud.
\]

Furthermore, the $(1, 3)$-array of $F^{-1}AF$ is

\[
d + 1/ \mid \begin{array}{c}
    c \\
    d
\end{array} \mid (lm - nk)(ct + dv) = -tc + (1 - v)d.
\]

For any $a, b \in \mathbb{Z}_p$, set

\[
\begin{align*}
    (1 - s)c - ud &= a \\
    -tc + (1 - v)d &= b
\end{align*}
\]

Then, there exist $c, d \in \mathbb{Z}_p$ satisfying $(\ast)$ if and only if

\[
\det(I - B) = \det(I - B^t) = \det \begin{bmatrix}
    1 - s & -u \\
    -t & 1 - v
\end{bmatrix} \neq 0,
\]

i.e., $I - B$ is regular. Note that $I - B$ is not regular if and only if 1 is one of the eigenvalues of $B$.

But, the representatives of conjugacy classes of $GL_2(\mathbb{Z}_p)$ are given as follows:

\[
C_\lambda = \begin{bmatrix}
    \lambda \\
    \lambda
\end{bmatrix} (1 \leq \lambda \leq p - 1),
D_{2,\lambda} = \begin{bmatrix}
    \lambda & 0 \\
    1 & \lambda
\end{bmatrix} (1 \leq \lambda \leq p - 1),
B_{\lambda,\tau} = \begin{bmatrix}
    \lambda \\
    \tau
\end{bmatrix} (1 \leq \lambda \neq \tau \leq p - 1), B_2^i (0 < i < p^2 - 1, i \neq 0 \text{ (mod } p + 1),
\]

where we select only one of $i$ and $i'$ such that $ip \equiv i'(mod p^2 - 1)$ for $B_2^i$ (see [11]). The cardinalities and the number of the first, second, etc., fourth type of conjugacy classes are as follows:

\[
1, (p - 1)(p + 1), p(p + 1), p(p - 1); p - 1, p - 1, \frac{1}{2}(p - 1)(p - 2), \frac{1}{2}p(p - 1).
\]

Then each of matrices $C_\lambda$, $D_{2,\lambda}$ $(2 \leq \lambda \leq p - 1)$ and $B_{\lambda,\tau} (2 \leq \lambda \neq \tau \leq p - 1)$ does not contain 1 as its eigenvalue. By Lemma 5 of [20], each $B_2^i (0 < i < p^2 - 1, i \neq 0 \text{ (mod } p + 1)$ does not contain 1 as its eigenvalue. Thus,

\[
F^{-1}AF = \begin{bmatrix}
    1 & a & b \\
    0 & 0 & B
\end{bmatrix} \text{ for any } a, b \in \mathbb{Z}_p,
\]

where $B = C_\lambda, D_{2,\lambda}, B_{\lambda,\tau}, B_2^i$. Therefore the following four matrices are given as the representatives of conjugacy classes of $GL_2(\mathbb{Z}_p)$:

\[
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & \lambda & 0 \\
    0 & 0 & \lambda
\end{bmatrix} (2 \leq \lambda \leq p - 1), \begin{bmatrix}
    1 & 0 \\
    0 & 0
\end{bmatrix} (2 \leq \lambda \leq p - 1), \begin{bmatrix}
    0 & 1 \\
    1 & 0
\end{bmatrix} (2 \leq \lambda \neq \tau \leq p - 1), \begin{bmatrix}
    1 & 0 \\
    0 & 0
\end{bmatrix} (2 \leq \lambda \neq \tau \leq p - 1).
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \tau
\end{bmatrix} (2 \leq \lambda \neq \tau \leq p-1), \quad \begin{bmatrix}
1 \\
B_2^i
\end{bmatrix} (0 < i < p^2 - 1, i \not\equiv 0 \pmod{p+1}).
\]

The cardinalities and the number of the above four types of conjugacy classes are as follows:

\[p^2, p^2(p-1)(p+1), p^3(p+1), p^3(p-1); p-2, p-2, \frac{1}{2}(p-2)(p-3), \frac{1}{2}p(p-1).\]

The representatives of conjugacy classes of \(GL_2(\mathbb{Z}_p)\) which contain 1 as eigenvalues are given as follows:

\[
C_1 = I_2 = \begin{bmatrix}
1 \\
1
\end{bmatrix}, D_{2,1} = \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}, B_{1,\lambda} = \begin{bmatrix}
1 \\
\lambda
\end{bmatrix} (2 \leq \lambda \leq p-1).
\]

Case 1: \(B = C_1\).

\[
F^{-1}I_3F = I_3 \text{ for any } F \in GL_2(\mathbb{Z}_p).
\]

Next, let

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, F = \begin{bmatrix}
1 & c & d \\
0 & D
\end{bmatrix}, D = \begin{bmatrix}
a & b \\
m & n
\end{bmatrix} (an - bm \neq 0).
\]

Then we have

\[
F^{-1}AF = \begin{bmatrix}
1 & a & b \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

for any \(a, b \in \mathbb{Z}_p\).

Therefore the following two matrices are given as the representatives of conjugacy classes of \(GL_2(\mathbb{Z}_p)\):

\[
I_3 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The cardinalities and the number of the above two types of conjugacy classes are as follows:

\[1, p^2 - 1; 1, 1.\]

Case 2: \(B = D_{2,1}\).

Then, let

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}, F = \begin{bmatrix}
1 & c & -a \\
0 & D
\end{bmatrix}, D = \begin{bmatrix}
k & 0 \\
m & k
\end{bmatrix} (k \neq 0).
\]

Then we have

\[
F^{-1}AF = \begin{bmatrix}
1 & a & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]
for any \( a \in \mathbb{Z}_p \).

Next, let

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix},
F = \begin{bmatrix}
1 & c & -a + m \\
0 & D & 0 \\
m & b & m
\end{bmatrix},
D = \begin{bmatrix}
b & 0 \\
m & b
\end{bmatrix} (b \neq 0).
\]

Then we have

\[
F^{-1}AF = \begin{bmatrix}
1 & a & b \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

for any \( a \in \mathbb{Z}_p \) and \( b \in \mathbb{Z}_p^* \).

Therefore the following two matrices are given as the representatives of conjugacy classes of \( GL_2(\mathbb{Z}_p)_e \):

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

The cardinalities and the number of the above two types of conjugacy classes are as follows:

\[
p(p - 1)(p + 1), p(p - 1)^2(p + 1); 1, 1.
\]

Case 3: \( B = B_{1, \lambda} \).

Then, let

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \lambda
\end{bmatrix},
F = \begin{bmatrix}
1 & c & (1 - \lambda)^{-1}b \\
0 & D & 0 \\
0 & 0 & n
\end{bmatrix},
D = \begin{bmatrix}
k & 0 \\
0 & n
\end{bmatrix} (kn \neq 0).
\]

Then we have

\[
F^{-1}AF = \begin{bmatrix}
1 & 0 & b \\
0 & 1 & 0 \\
0 & 0 & \lambda
\end{bmatrix}
\]

for any \( b \in \mathbb{Z}_p \).

Next, let

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & \lambda
\end{bmatrix},
F = \begin{bmatrix}
1 & c & b(1 - \lambda)^{-1} \\
0 & D & 0 \\
0 & 0 & n
\end{bmatrix},
D = \begin{bmatrix}
a & 0 \\
0 & n
\end{bmatrix} (an \neq 0).
\]

Then we have

\[
F^{-1}AF = \begin{bmatrix}
1 & a & b \\
0 & 1 & 0 \\
0 & 0 & \lambda
\end{bmatrix}
\]

for any \( a \in \mathbb{Z}_p^* \) and \( b \in \mathbb{Z}_p \).
Therefore the following two matrices are given as the representatives of conjugacy classes of $GL_2(\mathbb{Z}_p)$:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \lambda
\end{bmatrix}
\quad (2 \leq \lambda \leq p - 1),
\quad \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & \lambda
\end{bmatrix}
\quad (2 \leq \lambda \leq p - 1).
\]

The cardinalities and the number of the above two types of conjugacy classes are as follows:

\[p^2(p+1), p^2(p-1)(p+1); p-2, p-2.\]

Q.E.D.

References


[23] (論文：対称有向グラフのある cyclic abelian covers の同型について II 著者：みずの ひろふみ，明星大学情報学部 さとう いわお，小山高専 受付：平成 12 年 12 月 22 日)